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IFRS 9, Stress Testing, ICAAP:  
a comprehensive framework for PD calculation  

Carlo Toffano, Francesco Nisi and Lorenzo Maurri  

Abstract: In order to fulfil all the different requirements coming from competent authorities for different regulatory processes, Financial Institutions are asked to calculate probabilities of default (PD) in several ways. For instance, a stressed Point in Time PD is required for stress-testing purposes to calculate impairment losses, a Through the Cycle PD is needed for RWA calculation while forward looking lifetime PDs are needed for the new IFRS 9 accounting standards compliance. This plurality gives rise to the need for synergism among the different PD calculation methods in order to minimize costs related to the PD model development and maintenance. Furthermore, synergies are required to both keep coherence among the different PD calculation methodologies and address those requirements pressing for high integration between different regulatory frameworks (as can be seen from latest EBA guidelines on the 2018 EU-wide Stress Test that will include IFRS 9 methodology).  

To this end, in this paper we propose a comprehensive framework addressing some of the main problems arising in developing PDs for regulatory purposes such as how to: derive Point in Time PDs, include macroeconomic scenario, develop forward looking transition matrix, calculate stressed Point in time PDs and compute forward looking lifetime PDs for stress testing and IFRS 9 purposes. Our Markov-based cumulative PD model has been actually implemented. We found that variations in both the long-term convergence transition matrix and convergence’s parameter, provide the required flexibility in the cumulative PD curves to guarantee high representative accuracy of historical data. We believe that such a framework will provide practitioners with a powerful tool useful to minimize the models’ maintenance costs, facilitate the supervisory review process, and guarantee the coherence with currently used PD models, IT architecture and managerial processes. From a practical point of view, we propose 10 operational steps in the same methodological framework useful to address many of the problems faced by banks when accomplishing with IFRS 9, Stress Testing and ICAAP requirements.  

Keywords: Credit Risk, IFRS 9, Stress Testing, ICAAP, PD, Merton, Markov, migration matrices.  

1. Introduction  

Starting from Basel II introduction in the first decade of 2000, steady-state, stable parameters have been required to banks for the calculation of risk-weighted assets (European Parliament, 2013). The objective of the authorities was to produce stable capital requirements, so avoiding procyclicality of regulatory capital requirements (see for example Repullo et al. (2009), Rikkers and Thibeault (2008), Marshall (2009), Gordy and Howells (2006) and BCBS (2010) for details about procyclicality of regulatory capital requirements). As stated by Carlehed and Petrov (2012) checking the assumptions used for deriving the Basel II risk-weighted asset (RWA) formulas (see, for example, Aas (2005) and Hamerle et al. (2003)), the average of “Point-in-Time” (PIT) PDs should be used in RWA formulas.  

Those parameters are usually known as “Through-the-Cycle” (TTC) average PDs and represent an “average” or prudential PD over the entire business cycle. Basel risk parameters are also characterized by margins of conservatism, so they can be interpreted as the bank’s best estimate plus a conservative add-on (Van Gestel, 2014).
Later, regulatory or accounting evolution moved from prudential long-run estimates towards pure best estimates approaches trying to capture the actual underlying risk of financial assets at a specific point in time (Aguais, 2015). This extended approach is reflected both in provisioning rules (IFRS 9) and in stress testing exercises (regulatory Stress Test, SREP stress test).

As a result of this evolution, now banks are required to provide probabilities of default (PD) in several forms. Versions vary from stressed/unstressed PD to change in the rating philosophy itself (‘point in time’ vs ‘through the cycle’ parameters) and 12 months or lifetime PD.

In particular, the newly introduced IFRS 9 accounting standard involves PIT parameters including also a forward-looking perspective (Ankarath et al., 2010; BCBS, 2015; IASB, 2014). Under IFRS 9, hence, estimates of PD will change following the economic cycle. In particular, as regulatory PDs reflect longer-term trends in PD behavior (whose estimates are less sensitive to changes in economic conditions), PIT PDs will be generally lower than TTC PDs in periods of positive macroeconomic conditions and higher during macroeconomic downturns.

Similarly, migrations of exposures and default rates for EBA Stress Test exercise depend on a point in time within the business cycle. As required by European Banking Authority guidelines, point-in-time risk parameters (PD PIT and LGD PIT) should be forward-looking projections of default rates and loss rates, and capture current trends in the business cycle. In contrast to through-the-cycle parameters they should not be business cycle neutral. PD PIT and LGD PIT should be used for all credit risk related calculations except RWA under both the baseline and the adverse scenario (EBA, 2016a,b).

Finally, Internal Capital Adequacy Assessment Process (ICAAP and SREP processes) asks for the inclusion of stress analyses based on the regulatory methodologies (EBA, 2016a,b) and internal stress test analysis, whereas IRB approach validation ask for both “Through the Cycle” (TTC) PDs and the availability of a stress testing framework in use.

The characteristic to be conditional to the forward economic cycle leads to the need for satellite models for different purposes.

It is clear from what above that there is a strong similarity of concepts required by accounting provisions calculations, stress testing and internal capital assessment, which rises the need for synergism between efforts spent for PD calculations. Moreover, latest EBA methodological guidelines on EU-wide Stress Test (EBA, 2017) explicitly includes the IFRS 9 provisioning impact and the use of “lifetime and scenario dependent probability of default”, so providing a regulatory incentive for an immediate integration between different frameworks.

This means that while financial institutions generally have TTC risk parameters developed for risk-weighted assets purposes, they are asked to provide PIT risk parameters for different regulatory requirements (IFRS 9, Stress Test, ICAAP), too. Those regulations do not provide any specific guidance on how to adjust TTC PD to PIT PD, but a consistent and transparent framework bridging the two types of measures is actually needed for banks.

The process is complex, and there are two points to be stressed to obtain synergies: first, Rating models developed for IRB purposes might be a good starting point, provided they can be reconciled to IFRS 9 PDs, and second, there is a clear similarity between PIT regulatory requirements for IFRS 9, Stress Test and ICAAP for estimates characterized by high responsiveness to current and forward looking macroeconomic scenarios.

In this context, the challenge faced by banks is to produce regulatory compliant as well as coherent PD estimates while minimizing the amount of resources devoted to their development and maintenance. In our opinion, the best way to accomplish this target, even from a regulatory compliance point of view, is to leverage on risk parameters developed for RWA purposes (Basel II regulation) and to adapt them according to the specific needs.

Our paper proposes a simple and comprehensive framework for PD calculation aimed at satisfying the need for synergism between efforts made for PD calculations according to different regulatory requirements. It would also enable banks to easily develop Stress Test framework coherently with latest EBA guidelines (EBA, 2017) for what concerns PDs estimation and IFRS 9. Our proposed framework is based on the widespread use of Merton model (Gordy 2003; Merton 1974;
Miu and Ozdemir 2009; Vasicek 2002; Yang and Du 2015, 2016), thus it is founded on sound methodology while remaining simple enough to be clearly communicated during the supervisory review process.

Moreover, we implemented our Markov-based cumulative PD model on an illustrative rating migrations’ sample in order to assess its fitting properties. We found that variations in both the long-term convergence transition matrix and convergence’s pace parameter provide the required flexibility in the cumulative PD curves so guaranteeing high representative accuracy.

The rest of the paper is organized as follows. In Section 2, we present the theoretical development framework and discuss the problems addressed in each specific subsection. Section 3 concludes.

2. Theoretical framework

This section describes the details of our theoretical framework. It is divided into ten subsections, one for each of the main problems faced by Banks in accomplishing regulatory requirements for PD calculations.

Our methodology is based on the assumption that internal rating models are already in place within the Bank. Starting from such internal models banks have to:

1. obtain PiT PDs and long term PDs from already existing PDs;
2. link default dynamics to the macroeconomic cycle;
3. include scenario dependency into bank portfolio pit PDs;
4. calculate migration probabilities scenario dependent;
5. converge to long term equilibrium;
6. guarantee coherence between lifetime PDs and stressed PDs;
7. forecast scenario dependent PDs and long-term PDs;
8. forecast scenario dependent and long-term migration probabilities;
9. derive scenario dependent lifetime PDs;
10. assess the fitting of the estimated PD curves.

step1: obtain PiT PDs and long term PDs from already existing PDs

The first problem to be addressed is the derivation of PIT PDs required by both the newly introduced accounting standards and stress test exercises.

Generally speaking, there are several methods that can be used to transform TTC PDs into PIT PDs (e.g. by calibrating models on shorter and more recent time horizons, or by deriving PDs by empirical default frequency analyses or using a Merton framework as in Carlehed and Petrov (2012)) and all of them are perfectly suitable depending on data available.

But beyond the methodology used to derive the PD PIT, how can we formally represent them since they are conditioned to current business and economic cycle?

Retrieving the Merton model for conditional PD modeling (Gordy 2003; Merton 1974; Miu and Ozdemir 2009; Vasicek 2002; Yang and Du 2015, 2016), we can define probability of default as the probability that the creditworthiness index of the firm falls below a certain threshold.

Obligor-specific probabilities of default can then be expressed in terms of macroeconomic factors, as a combination of obligor-specific, cluster-specific and portfolio-specific terms.

\[ PD_{i(t)} = Pr\left( y_{i(t)} \leq c_i \right) \]

Where:

\( i \) = i-th company
\( t \) = time instant
\[ y_{i(t)} = \alpha_i z_{k(t)} + \sqrt{1 - \alpha_i^2} \epsilon_{i(t)} \]

\[ z_{k(t)} = \sum_{j=1}^{J} \beta_{kj} X_{j(t)} + \eta_{k(t)} \]

\( k = \) portfolio cluster "k" for systematic component

\( c_i = \) default threshold

All random variables are assumed to be distributed as standard normal \((N(0,1))\), with the exception of macroeconomic factors which are assumed to be normally distributed with 0 mean and given correlation matrix. In other words, the obligor-specific creditworthiness \( y_{i(t)} \) is a function of:

- **Systematic component** \( z_{k(t)} \): representing the economic system effect;
- **Idiosyncratic component** \( \epsilon_{i(t)} \): representing the obligor-specific effect;
- **Single firm correlation to systematic components** \( \alpha_i \).

The systematic component \( z_{k(t)} \) is in turn a function of:

- **Macroeconomic component** \( X_{j(t)} \): representing the macroeconomic factors effect;
- **Cluster-specific component** \( \eta_{k(t)} \): representing an idiosyncratic cluster-specific term effect;
- **Factor loadings** \( \beta = [\beta_{k1}, \beta_{k2}, ..., \beta_{kJ}] \) and factor to factor correlation structure \((\Sigma)\).

So, probability of default conditional to current cycle can be expressed as:

\[ PD_{i(t)} = Pr\left(y_{i(t)} \leq c_i\right) = Pr\left(\alpha_i z_{k(t)} + \sqrt{1 - \alpha_i^2} \epsilon_{i(t)} \leq c_i\right) = Pr\left(\sqrt{1 - \alpha_i^2} \epsilon_{it} \leq c_i - \alpha_i z_{k(t)} \right) = Pr\left(\epsilon_{it} \leq \frac{c_i - \alpha_i z_{k(t)}}{\sqrt{1 - \alpha_i^2}}\right) \]

To design the lifetime forward looking path of PD for IFRS9 purposes, banks are also engaged to provide long term PDs and to lead a convergence of the path in the long term. The problem can be faced, as usual, in several ways. Ideally, long term probabilities can be derived from time series long enough to cover the whole economic cycle, but the availability of such data in banks is questionable. Alternative approaches can be implemented like the one proposed by Rubtsov and Petrov (2016), the TTC adjustment proposed by the UK Financial Services Authority (FSA, 2007) or starting from PIT matrices and sterilize them from the economic cycle effect by a reverse Merton approach.

So, conditional to a long term equilibrium of systematic factor, the probabilities of default can be expressed as:

\[ PD_{i(t \rightarrow \infty)} = Pr\left(\epsilon_{i(t \rightarrow \infty)} \leq \frac{c_i - \alpha_i z_{k(t \rightarrow \infty)}}{\sqrt{1 - \alpha_i^2}}\right) \]

Due to the need of synthesis, we will not describe here this well known method but highlight that at the end of this task we have a formulation with a PIT PD and a long run PD that will be used in our framework.

**step2: link default dynamics to the macroeconomic cycle**
Stress testing exercises and IFRS9 require the construction of a satellite model linking the default risk to macroeconomic factors dynamics guaranteeing at the same time extreme sensitivity to the scenario provided by supervision authorities (see EBA Stress Test, 2016a,b).

Hence, the objective of the satellite model is to define a synthetic credit risk factor correlated with the probability of default dynamics (for example, a transformation of the empirical default rate for different portfolio clusters) and link this synthetic factor to macroeconomic factors.

Satellite model typically available may be based:

- on level variables (in case of stationary time series) so that the model assumes the form:

\[ z_k(t) = \sum_{j=1}^{f} \beta_{0,kj} X_j(t) + \eta_{k(t)} \]

- on first differences: in case of macroeconomic factors expressed as logarithms, they represent percentage differences which are typically used for macroeconomic scenario definition.

\[ \Delta z_k(t) = \sum_{j=1}^{f} \beta_{0,kj} \Delta X_j(t) + \eta_{k(t)} \]

- on multiple regression through an Error Correction Models (ECM) or some other dynamics models. The ECM model is a two steps model (Engle and Granger, 1987), in which the first regression defines the equilibrium relation, the variables are in levels, and the second model aims to catch the point in time variation in difference integrated with the equilibrium relation. So, the model consists of components in level and components in differences as follows.

  **Short-term equation:**

\[ \Delta z_k(t) = \sum_{j=1}^{f} \beta_{0,kj} \Delta X_j(t) + (\gamma - 1) \ast v_{k(t-1)} + \eta_t \]

  **Equilibrium equation:**

\[ z_k(t) = \sum_{j=1}^{f} \beta_{0,kj} X_j(t) + v_{k(t)} \]

Benefits of ECM include the ability to catch the dynamics of the PD point in time with respect to macroeconomic factors variations in the short term as well as in the long term, with mean reverting properties. Moreover, the ability to treat the non-stationary macroeconomic time series throughout the property of cointegration defined in the equilibrium equation.

**step3: include scenario dependency into bank portfolio pit PDs**

Starting from the result obtained in step1, we can arrange formulas to derive the default threshold \( c_t \) as a function of Probability of Default and systematic factor index.
The same equation holds also for \( t - 1 \); we can obtain \( c_i \) as a function of \( PD_{i(t-1)} \) and \( z_{k(t-1)} \).

Once obtained the fixed default threshold, it can be substituted into the Merton formula to derive future conditional probabilities of default (\( PD_{i(t)}, PD_{i(t+1)}, \ldots \)) from the available PDs at time \( t - 1 \), as follows:

\[
PD_{i(t)} = \Phi \left( \frac{c_i - \alpha_i z_{k(t)}}{\sqrt{1 - \alpha_i^2}} \right)
\]

\[
\Phi^{-1}(PD_{i(t)}) = \frac{c_i - \alpha_i z_{k(t)}}{\sqrt{1 - \alpha_i^2}}
\]

\[
c_i = \sqrt{1 - \alpha_i^2} \Phi^{-1}(PD_{i(t)}) + \alpha_i z_{k(t)}
\]

So, for each “\( t \)”, future probabilities of default can be derived iteratively from previous PD and expected variations in the macroeconomic scenario. In the case above, PD at time \( t \) can be obtained from PD at time \( t - 1 \) plus the expectation of the change in the macroeconomic scenario from \( t - 1 \) to \( t \).

**step 4: calculate migration probabilities scenario dependent**

In case of rating models availability, EBA Stress Testing and IFRS 9 framework (in the case of use of a Markov inhomogeneous approach) request portfolio risk dynamic be led by transition matrices conditioned by the macroeconomic scenario (3 years “baseline” and “adverse” scenario are provided for the EBA Stress Test exercise).

The passages developed at PD level in the previous sections can be easily extended to the migration matrix case as follows. Suppose to have an obligor with rating “\( r \)”, the migration probabilities on a generic rating class “\( m \)” depend on rating class threshold as follows

\[
q_{r,m,t} = Pr(c_{r,m} \leq y_{i,r,t} \leq c_{r,m+1})
\]

Where \( q_{r,m,t} \) is the rating migration probability (for the \( i \)th obligor belonging to rating class “\( r \)” (at time \( t \)) to a rating class bounded by threshold “\( m \)” and “\( m + 1 \)”.

Graphically, it represents the area under the PD density curve for an asset with initial rating class \( r \) as represented in the following graph.
The migration probability from rating class “r” is given by the probability that the creditworthiness factor lies between two fixed thresholds. From the previous formula it can be shown that:

\[
q_{r,m,t} = \Phi \left( \frac{c_{r,m+1} - \alpha_i z_k(t)}{\sqrt{1 - \alpha_i^2}} \right) - \Phi \left( \frac{c_{r,m} - \alpha_i z_k(t)}{\sqrt{1 - \alpha_i^2}} \right)
\]

So, the migration probability \( q_{r,m,t} \) can be expressed as the difference between two cumulative standard normal distributions \( (P_{r,m,t}) \). It is worth noticing, similarly to step3, that the fixed threshold \( c_{r,m} \) can be easily derived by inverting the \( P_{r,m,t} \) formula obtaining:

\[
c_{r,m} = \Phi^{-1}(P_{r,m,t}) \sqrt{1 - \alpha_i^2} + \alpha_i z_k(t)
\]

The same equation holds also for \( m + 1 \) obtaining \( c_{r,m+1} \) as a function of \( P_{r,m+1,t} \) and \( z_k(t) \) and for “t-1” as a function of \( P_{r,m+1,t-1} \) and \( z_k(t-1) \).

Given the two thresholds, it is possible to derive migration probabilities at time \( t \) as \( q_{r,m,t} \); by substituting \( c_{r,m+1} \) and \( c_{r,m} \) known from the transition matrix in time \( t - 1 \), we get:

\[
q_{r,m,t} = \Phi \left[ \frac{\Phi^{-1}(P_{r,m+1,t-1}) \sqrt{1 - \alpha_i^2} + \alpha_i z_k(t-1) - \alpha_i z_k(t)}{\sqrt{1 - \alpha_i^2}} \right]
- \Phi \left[ \frac{\Phi^{-1}(P_{r,m,t-1}) \sqrt{1 - \alpha_i^2} + \alpha_i z_k(t-1) - \alpha_i z_k(t)}{\sqrt{1 - \alpha_i^2}} \right] =
\]

\[
= \Phi \left[ \Phi^{-1}(P_{r,m+1,t-1}) - \frac{\alpha_i}{\sqrt{1 - \alpha_i^2}} (\Delta z_k(t)) \right] - \Phi \left[ \Phi^{-1}(P_{r,m,t-1}) - \frac{\alpha_i}{\sqrt{1 - \alpha_i^2}} (\Delta z_k(t)) \right]
\]

Hence, the migration probability at time \( t \) depends on the migration probability at time \( t - 1 \) plus the change in macroeconomic cycle from \( t - 1 \) to \( t \) \( (\Delta z_k(t)) \).

What we obtained is a relation that links the migration probabilities of the PIT transition matrix at time \( t \) to the macroeconomic forecasts in \( t \) provided by the satellite model seen in step2 and the rules to use them in order to forecast the migration probabilities.
step 5: converge to long term equilibrium

The typical need in lifetime PD calculation (for IFRS9 lifetime expected losses) is to hypothesize long-term convergence of “future” PDs; in our framework this implies a convergence in systematic factor z that conditions the future PDs.

We show how to include long-run convergence of the systematic factor \( z_{k(t+l)} \) given the hypothesis in the previous paragraph of an ECM dynamics, as:

\[
\Delta z_{k(t+l)} = \sum_{j=1}^{l} \beta_{0,kj} \Delta X_{j(t+l)} + (\gamma - 1)(z_{k(t+l-1)} - \sum_{j=1}^{l} k_{0,kj} X_{j(t+l-1)}) + \eta_{k(t+l)}
\]

where \( l \) = number of steps ahead (lead) in forecast

We can assume that, in the long term, macroeconomic factors will converge towards a steady state (unless there are robust alternative hypotheses). This means that \( z \) score will tend to its long run equilibrium \( \bar{z}_k \) when \( l \rightarrow \infty \).

So, if \( \sum_{j=1}^{l} k_{0,kj} X_{j(t+l-1)} = \bar{z}_k \) then we can express \( \Delta z_{k(t+l)} \) as:

\[
\Delta z_{k(t+l)} = \sum_{j=1}^{l} \beta_{0,kj} \Delta X_{j(t+l)} + (\gamma - 1)(z_{k(t+l-1)} - \bar{z}_k) + \eta_{k(t+l)}
\]

That is, in the long run the variations in the \( z \) scores (\( \Delta z_{k(t+l)} \)) can be expressed as a function of both the macroeconomic factors’ variations (\( \Delta X_{j(t+l)} \)) and the deviation of the \( z \) score at \( t + l - 1 \) with respect to its long term average (\( z_{k(t+l-1)} - \bar{z}_k \)).

But do you think you could have the forecast of short term variations of macroeconomic factors (\( \Delta X_{j(t+l)} \)) in the long run? Are them available and robust?

step 6: guarantee coherence between lifetime PDs and stressed PDs

The limited availability of long term economic forecasts along with their poor accuracy (economic forecasts are typically available for the first 3 years as in EBA stress test exercises), implies a long term convergence hypothesis which is appropriate since default rate is hopefully stationary in mean.

We propose to neutralize the convergence process for the initial 3 years (in order to amplify the sensitivity of the models in short term exercises like stress test has required by regulators methodology) and, on the other hand, to neutralize first differences in the next years in order to maximize long term convergence (as hopefully expected for IFRS9 purposes).

Formally,

\[
\begin{align*}
\gamma &= 1 & \text{if } l = 1, 2, 3 \ldots \text{ years ahead in forecast} \\
\sum_{j=1}^{l} \beta_{0,kj} \Delta X_{j(t+l)} &= 0 & \text{if } l = 4, 5, 6, \ldots, N \text{ years ahead in forecast}
\end{align*}
\]
It follows from the assumed neutralization that:

\[
\begin{align*}
\Delta z_{k(t+l)} &= \frac{1}{\sum_{j=1}^l \beta_{0,kj} \Delta X_{j(t+l)} + \eta_{k(t+l)}} \quad \text{if } l = 1,2,3 \\
\Delta z_{k(t+l)} &= (\gamma - 1) (z_{k(t+l-1)} - \bar{z}_k) + \eta_{k(t+l)} \quad \text{if } l = 4,5,6,...,N
\end{align*}
\]

Actually, Stress testing exercises encourage the use of just the first expression with 3 years macroeconomic forecasts for stressed PD, whereas for IFRS 9 purposes (included in the stress testing framework EBA 2017) both expressions are useful to derive long term \(\Delta z\) estimates. The constraint in using only 3 years short term variations of macroeconomic factors (\(\Delta X_{j(t+l)}\)) is needed to avoid the generation of an unhampered process. Indeed, it is in the interest of the banks to govern the evolution of \(\Delta z\) in an economically meaningful way towards an equilibrium state.

Moreover, the gradual convergence process from the fourth year is conditioned by the path of the factor over the first three years that is scenario dependent.

Several different assumptions could be suitable, but we suggest this hypothesis since it has the relevant feature that allows for the coherence between Stress Testing and IFRS 9 frameworks and enable both processes.

**step7: forecast scenario dependent PDs and long-term PDs**

For “short term” exercises (3 years), given the assumption above, we can derive future probabilities of default (\(PD_{i(t+1)}, PD_{i(t+2)}, ...\)) from the available PDs at time \(t\), as follows:

\[
PD_{i(t+1)} = \Phi \left[ \Phi^{-1}(PD_{i(t+1-1)}) - \frac{a_i}{\sqrt{1-a_i^2}} \sum_{j=1}^l \beta_{0,kj} \Delta X_{j(t+l)} \right]
\]

So, for each “\(t\)”, future probabilities of default can be derived iteratively from previous PD and expected variations in the macroeconomic scenario.

We assumed in step5 that macroeconomic variables converge towards their steady state in the long term, then variations in the macroeconomic factors should be coherent to this convergence path. In this case, \(\Delta z_{k(t+l)}\) (the dynamic of the credit risk systematic factor) can be expressed with a convergence path to its long term value that depends on the speed of convergence; rearranging this formula we have that:

\[
z_k(t+1) = \gamma z_k(t+l-1) + (1 - \gamma) \bar{z}_k
\]

The z score at time \(t + l\) can be written as the weighted average of the z score at time \(t + l - 1\) and the long term score average (weight = \(\gamma\)). This result can be easily extended to derive each z score in the future starting from the previous score value and its long term equilibrium value.

For long term path we have to compute

\[
PD_{i(t+1)} = \Phi \left( \frac{c_i - a_i \bar{z}_{k(t+1)}}{\sqrt{1-a_i^2}} \right)
\]
That can be expressed as a function of $PD_{t(t+l-1)}$ and its long term convergence $\overline{PD}_t$:

$$PD_{t(t+l)} = \Phi[\gamma$ \Phi^{-1}(PD_{t(t+l-1)}) + (1 - \gamma) \Phi^{-1}(\overline{PD}_t)]$$

In other words, the probability of default at a generic time $t + l$ can be expressed as the cumulative normal transformation of the weighted average between PD at previous time ($PD_{t(t+l-1)}$) and long term convergence PD ($\overline{PD}_t$) in the normal distribution space. Convergence pace is governed by parameter $\gamma$ which can be either assigned to reach the long-run convergence in a specified time (e.g. we found that $\gamma = 0.5$ implies a convergence to the long-run in about 10 years), or estimated from the satellite model (see formulation in step 5).

**step 8: forecast scenario dependent and long-term migration probabilities**

As seen in step 4, also the rating transition probabilities depend on $z$ factor, so the procedure to derive their path to the long term value is analogous to PDs.

For short term exercises, 

$$q_{r,m,t+l} = \Phi\left[\frac{\alpha_i}{\sqrt{1-\alpha_i^2}} \sum_{j=1}^{f} \beta_{0,kj} \Delta X_{f(t+l)}\right] - \Phi\left[\frac{\alpha_i}{\sqrt{1-\alpha_i^2}} \sum_{j=1}^{f} \beta_{0,kj} \Delta X_{f(t+l)}\right]$$

For long term path, by substituting the expression of $z_{k(t)}$ into $P_{r,m+1,t+l-1}$ formula and doing the same passages developed at PD level, we get:

$$q_{r,m,t+l} = \Phi[\gamma \Phi^{-1}(P_{r,m+1,t+l-1}) + (1 - \gamma) \Phi^{-1}(\overline{P}_{r,m+1})]$$

- $\Phi[\gamma \Phi^{-1}(P_{r,m,t+l-1}) + (1 - \gamma) \Phi^{-1}(\overline{P}_{r,m})]$

That is, the migration probabilities in the future are equal to the normal transformation of the weighted average between the transition probability $t + l - 1$ and the long term transition probability both in the distribution space.

**step 9: derive scenario dependent lifetime PDs**

Once obtained the point in time scenario dependent PDs and migrations for each of the future years using the methodology described in the previous sections, it is easy to derive the forward looking cumulative PD curve by using an inhomogeneous Markov chain approach (see for example Suhov and Kelbert, 2008); this is a need in order to calculate the lifetime probability of default required by IFRS9 framework.

Generally speaking, according to the Markov approach, the scenario dependent cumulative transition matrix is given by the product of scenario dependent transition matrices a time $t, t + 1, ..., t + l$. 

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The point in time future transition matrix $M$ at time $t + l$ is scenario dependent and will be built by using all the parameters obtained with the formulas explained in the steps above.

To comply with regulatory requirements for stress testing, ICAAP banks should use $M_{t+l}$ scenario dependent to migrate exposure in the exercise horizon between rating classes and default, whereas for IFRS9 banks should use $MC_{t+l}$ extracting the points related to expiring date of the instrument; all the inputs we need are the long term equilibrium transition matrix, a starting “point in time” transition matrix and a satellite model.

**step10: assess the fitting of the estimated PD curves**

A relevant feature of the proposed methodology is the flexibility of the resulting PD curves with respect to variations in the convergence parameter $\gamma$ and in the long-term convergence matrix. This means that financial institutions could simply work on those values to maximize the estimated curves’ fitting to the actual default curves.

However, even if we are aware that large part of financial institutions aim at maximizing PD curves’ fitting, it is our opinion that such an objective is not actually needed, because of the regulatory purpose (see, for example, IFRS 9 regulation) of building cumulative curves which are forward looking and not backward looking. In other words, past default curves’ shape should not be the target of the current forward looking modeling if the path implied into the future PIT matrices and the long term matrix defines different-shaped curves. Differences between the estimated curves and the past observed curves represent just the result of different expectations about the future macroeconomic scenario.

3. Conclusions

A comprehensive framework for PIT, stressed and lifetime PDs calculation was proposed in this paper. We have shown how banks could accomplish several different regulatory requirements through a unique framework promoting coherence between methodological frameworks, results and enhancing synergies in estimation work.

In particular, we have shown how banks could proceed by implementing the following 10 operational steps:
- obtain PiT PDs and long term PDs from already existing PDs;
- link default dynamics to the macroeconomic cycle;
- include scenario dependency into bank portfolio pit PDs;
- calculate migration probabilities scenario dependent;
- converge to long term equilibrium;
- guarantee coherence between lifetime PDs and stressed PDs;
- forecast scenario dependent PDs and long-term PDs;
- forecast scenario dependent and long-term migration probabilities;
- derive scenario dependent lifetime PDs;
- assess the fitting of the estimated PD curves.

To comply with regulatory requirements for Stress Testing, ICAAP banks should implement steps that we called “short term” in this paper generating scenario dependency of PDs and migration matrices, whereas for IFRS9 compliance banks should implement all the steps described till the calculation of “scenario dependent lifetime PDs” using also long term convergence in migration matrices and PDs.

Moreover, we argue that such an approach provides further relevant features, such as the flexibility in the resulting output and a sound methodological basis.

We deem that a broad, comprehensive PD framework is becoming more and more necessary for an efficient steering of banks because of the increasing number of different regulatory requirements (e.g. Pillar I and Pillar II of Basel regulation, accounting standards for provisioning purposes, etc.). We think that our approach will provide practitioners with a tool useful to minimize the models’ maintenance costs, facilitate the supervisory review process, and guarantee the coherence with currently used PD models, IT architecture and managerial processes.

The proposed framework is sensitive to variations in the convergence speed parameter and to the long run convergence matrix, providing banks a flexible modeling tool able to produce the expected PD term structures. On the other hand, this framework can build forward looking PD curves whose shape is not just the replication of past observed paths, but actually is the result of the expected macroeconomic scenario evolution (optimistic, baseline or adverse for example).

These considerations pave the way for the increasingly pressing theme of PD curves’ backtesting. As we mentioned earlier, the proposed model (as well as any other PD term structure model developed for similar purposes) cannot be tested by simply comparing estimated PD curves with past observed results, it should instead be tested by taking into account the effect of the actual and predicted macroeconomic scenario on large time horizons. We believe that this should be the base for future research on this field.
References


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